Vortex lattices as a key model for the Fractional Quantum Hall Effect . Published J.Phys.:Condens.Matter 21 (2009)405601

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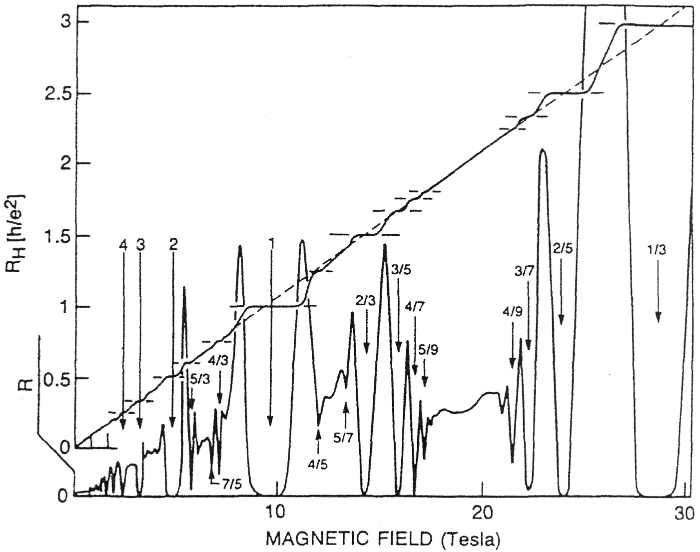
picture



R_H [h/e^2]
R
3
2.5
2
1.5
1
0.5
0
0
10
20
30
MAGNETIC FIELD (Tesla)
4
3
2
1
5/3
4/3
7/5
4/5
5/7
2/3
3/5
4/7
5/9
4/9
3/7
2/5
1/3

In spite of a lot of experimental and theoretical works devoted to IQHE K.von Klitzing(1980) and FQHE Tsui,Stoermer,Gossard (1982)the physical theory is far from being complete. The needed large value of magnetic field lead to the standard assumption that the ground state can be obtained by the projection of the states on the lowest Ll.(e.g . the famous Laughlin wave function).These states are degenerate and have zero average current which vanish on the distance of the order of magnetic length.

thermodynamic

If the states have some current one must calculate the thermodynamic energy involving corrections due to the work of the electric field on this current in the sample (see e.g. Landau,Lifshitz, Electrodynamic of continuous media)

$$\delta F = -\frac{1}{c}\int \mathbf{A}ext < \delta\mathbf{j} > dV$$

for thermodynamic energy at fixed external -vector potential . That is true or the 2DES where the external magnetic field can not be essentially changed by weak 2d currents.

hamiltonian

This equation can be integrated

$$F = H_i - \frac{1}{c}\int \mathbf{A}_{ext}\mathbf{j}d^2r$$

Where the "internal" energy is

$$H_i = \frac{\hbar^2}{2m_e}\int \psi^+_\sigma(-i\nabla + \frac{e}{c\hbar}\mathbf{A})^2\psi_\sigma d^2r +$$
$$+1/2\int V_c(\mathbf{r}-\mathbf{r}'\psi^+_\sigma(\mathbf{r}\psi^+_{\sigma'}(\mathbf{r}'\psi_{\sigma'}(\mathbf{r}')\psi_\sigma(\mathbf{r})d^2rd^2r'$$

Here I write Coulomb energy omitting for brevity the interaction with positive charged background. If the compensating charge density is equal to the density of the filled Ll the thermodynamic energy will be equal to the internal energy and the current correction vanish.

instability

Otherwise it is possible to have a current with $div\mathbf{J} = 0$ because of charge conservation. Let us consider an isolated vortex in a hypothetical uniform state constructed from the states of lLl. In the axisimmetrical gauge

$$\psi(\mathbf{r}) = \sum_{m>0} c_m \exp(-im\phi) R_m(r)$$

ϕ is a polar angle, $R_m(r) N_m r^m \exp(-\frac{r^2}{4l_B^2}$ The arising current is connected with the singularity in the phase due to the vortex at the origin , and the additional vector-potential in the current expression is $\mathbf{A}' = \frac{Kc\hbar}{r|e|}\mathbf{e}_\phi(r)$ where K is integer and $\mathbf{e}$ is the unit vector tangent to the circle.

correction

The change of the current correction is

$$\delta F = \frac{\hbar^2 K}{2m_e l_B^2} \int_0^R \sum_{m>0} R_m^2 < c_m^+ c_m > 2\pi r dr$$

At large distances from the origin the changes to the wave function are small and all calculations can be performed in the first order of the perturbation theory and the integral gives the total number of electrons. Therefore for the negative K this quantity decreases proportional to the square of the sample size. The calculation of the change in the inner energy proceeds in the same way

$$\delta H_i = \frac{\hbar^2}{2m_e l_B^2} \int_0^R \frac{K^2}{r^2} \sum_{m>0} R_m < c_m^+ c_m > 2\pi r dr$$

It grows only as a logarithm of the sample size. Thus the vortex formation gives the gain in the energy for large enough sample.

Therefore one must consider the vortex lattices. The structure of the vortex core depend on the model. I consider the case of the strong ferromagnet. In this case the magnetzation axis will be rotated in the sample by 2×2 matrix $U = U_z(\gamma)U_y(\beta)U_z(\alpha)$ depending on three Euler angles.

$$\begin{pmatrix} cos(\beta/2)\exp(\frac{i(\alpha+\gamma)}{2}) & sin(\beta/2)\exp(\frac{i(\alpha-\gamma)}{2}) \\ -sin(\beta/2)\exp(\frac{-i(\alpha-\gamma)}{2}) & cos(\beta/2)\exp(\frac{-i(\alpha+\gamma)}{2}) \end{pmatrix}$$

It is possible to treat U as a selfconsistent quantity establishing the correlations of electron states along the sample by means of the canonical transformation of electron spinors $\psi = U\chi, \psi^{+} = \chi^{+}U^{+}$

Any physical operator must be calculated in a new representation .The current operator reads

$$\mathbf{j} = \frac{\hbar e}{2m_e}[\chi^+(-i\nabla\chi + i\nabla\chi^+\chi + \frac{2e}{c\hbar}\mathbf{A} + 2\chi^+\Omega\chi]$$

,

$$\Omega = (-i)U^+\nabla U$$

The simple calculation gives

$$\Omega = (\nabla\alpha/2 + cos\beta\nabla\gamma/2)\sigma_z + (cos\alpha\nabla\beta/2 + sin\beta sin\alpha\nabla\gamma/2)\sigma_y +$$
$$+(-sin\alpha\nabla\beta/2 + sin\beta cos\alpha\nabla\gamma/2)\sigma_x$$

σ_i are Pauli matrices. Assuming strong ferromagnet I can neglect non diagonal terms.In order to construct non singular matrix with nontrivial topological properties I suppose that the angle α has vortex like singularites on some lattice and $\nabla\gamma = \nabla\alpha + \nabla\gamma'$ where $\nabla\gamma'$ is a regular periodic function; $\beta = \pi$ on the vortex lattice, ,$\beta = 0$ on the unit cell boundary and is periodic

hamiltonian2

That gives the current operator

$$\mathbf{j} = \frac{\hbar e}{2m_e}\left(-i\chi^+\nabla\chi + i\nabla\chi^+\chi + 2\chi^+(\frac{e}{c\hbar}\mathbf{A} + \right.$$

$$+|\frac{1+cos\beta}{2}\nabla\alpha + cos\beta\frac{\nabla\gamma'}{2})\chi$$

The calculation of the internal energy proceeds in the same way but there is an additional diagonal term due to the squares of Pauli matrices σ_x, σ_y .The thermodynamic energy acquires the form

$$F = \frac{\hbar^2}{2m}\int \chi^+\left((-i\nabla - \frac{e}{c\hbar}\mathbf{A} + \frac{1+cos\beta}{2}\nabla\alpha + cos\beta\nabla\gamma'/2)^2 + \right.$$

$$+1/4\nabla\beta)^2 + 1/4(sin^2\beta(\nabla\alpha + \nabla\gamma')^2\chi d^2r - \int \frac{1}{c}\mathbf{A}\mathbf{j}d^2r +$$

$$+\frac{1}{2}\int V_c(\mathbf{r}-\mathbf{r}')\chi^+(\mathbf{r})\chi^+(\mathbf{r}')\chi(\mathbf{r}')\chi(\mathbf{r})d^2rd^2r'$$

poles

By assumption $\nabla\alpha$ has a periodic lattice of singular points (a system of poles in every unit cell).The divergence of the energy in these poles is eliminated by the choice of β.The realization may be achieved e.g. by the use of $\nabla\alpha = (Re\zeta, Im\zeta)$ where ζ is Weierstrass *zeta* function

$$\zeta = 1/z + \sum_{n,n'} | \left(1/(z - T_{nn'}) + 1/T_{nn'} + z/T^2_{nn'}\right)$$

$z = x + iy, T_{nn'}$ are complex periods.Or it is a sum of ζ functions with the shifted singularities. In any case it has the property $\zeta(z+\tau) = \zeta(z) + \delta(\tau)$

mtranslations

If we assume $\nabla\gamma = \nabla\alpha$,$\nabla\gamma' = 0$ in zero unit cell then after the translation to the another cell $\mathbf{r} \rightarrow \mathbf{r} + \vec{\tau}$, $\nabla\gamma(\mathbf{r} + \vec{\tau}) = \nabla\gamma(\mathbf{r}) + \vec{\delta}(\vec{\tau})$ by the property of *zeta*- function. This change can be eliminated by the proper choice of $\nabla\gamma' + \vec{\delta}(\vec{\tau}) = 0$. In the same way the change in $(1 + cos\beta)\nabla\alpha(\mathbf{r} + \vec{\tau}) - \frac{e}{c\hbar}\mathbf{A}$ can be eliminated by the change of the pase for χ :$\nabla\phi - \frac{e}{c\hbar}\mathbf{A}(\vec{\tau}) + \nabla\gamma'/2 = 0$ Thus the effective gamiltonian is invariant under the combined translation, the additional rotation on γ' and the change of the phase, i.e. magnetic translation.

electron density

It is well known that the finite representations with the symple band structure are valid only for the rational values of the total flux

$$BS + K\Phi_0 = \frac{l}{N}\Phi_0$$

through unit cell area S, $-K$is the number of the vortices in the unit cell, l and N are integers. The proper representations of the magnetic translation group are well known and are enumerated by quasimomentum and for the filled band corresponds to one electron per the unit cell of the vortex lattice with $NM_1 \times NM_2$ unit cells. Therefore the filled bands with the gaps on the boundaries may exsist only at the electron density

$$n_e = \frac{1}{S} = \frac{N}{l - NK}$$

All measured fractures correspond to the validity of this formulae for the lattices with one or or two unit vortices in unit cell.

Excitations

It is possible to use Landau argumentation for the explanation of observed Hall conductance in a weak applied electric field going to the system moving with the drift velocity, where electric field is zero. The electron fluid can change it's velocity only by the birth of excitation. These excitation are neutral because they consists from an electron on the higher empty band and a hole on the filled band.Therefore they have a conserved momentum in spite of magnetic field. The excitation energy will be

$$\epsilon(p) + \mathbf{pv}(\mathbf{E})$$

if this quantity is positive, the birth will be impossible at low enough temperatures, and the Hall current is proportional to the electron density.

The set of the observed electron densities depend on the value of the gaps, temperatures, and sample purity. To take into account these numerous factors is a difficult task. But it is easy to solve the opposite problem- to find the vortex lattices corresponding to the observed densities using the expression for the electron density $n_e = \frac{B_0}{\Phi_0}\frac{N}{l-NK}$.

The observed fractions are given by the following tables.

$K = -2, \quad l = 1$

N	1	2	3	-5	-2	-3	-4	4	∞
ν	$\frac{1}{3}$	$\frac{2}{5}$	$\frac{3}{7}$	$\frac{5}{9}$	$\frac{2}{3}$	$\frac{3}{5}$	$\frac{4}{7}$	$\frac{4}{9}$	$\frac{1}{2}$

That fractions correspond to the celebrated Jain's rule. Especially must be noted the half filling of the Ll corresponding to $limN \rightarrow \infty$ and the effective flux equal to zero. In this case we have an ordinary group of the translations with a normal band structure. This state can be achieved by the increasing of electron density $N > 0$ as well as by decreasing $N < 0$. This circumstance must create the symmetry vanishing of the gap for electron and holes.

Other observed fractions correspond to the table

$K = -1, \;\; l = 1$

N	-4	4	2
ν	$\frac{4}{3}$	$\frac{4}{5}$	$\frac{2}{3}$

where one have the double of the fraction $2/3$, and also

$K = -1, \;\; l = 2$

N	-7	-5	5	2
ν	$\frac{7}{5}$	$\frac{5}{3}$	$\frac{5}{7}$	$\frac{1}{2}$

here one have not observed double of the fraction $1/2$ with the gap. The exclusion of the doubles requires extensive numerical calculations.

All observed fractions correspond to vortex lattices with 1 or 2 flux quanta per unit cell.